Remarks on BPS bound state "decay"

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In $\mathcal{N}=2$ super Yang-Mills theory the charges of a Bogomol'nyi-Prasad-Sommerfield (BPS) state shift under the transition to a dual description of the theory. In particular, in the theory with matter duality, transformations may convert a bound state to an unbound one as predicted by Seiberg and Witten from considerations of the monodromies around the moduli space singularities. The physical mechanism of such behavior on the semi-classical level can be established explicitly through the consideration of soliton-fermion classical field configurations. The problem reduces to the investigation of the fermion spectrum in a slowly varying background monopole field. The behavior of the solutions to classical equations allows one to observe the BPS bound state "decay" *in vivo*.

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I. INTRODUCTION

One of the greatest recent successes of supersymmetry (SUSY) applications was the evaluation of the low energy effective action for $\mathcal{N}=2$ supersymmetric Yang-Mills theories. This became possible because supersymmetry imposed strong restrictions on the theory and, in particular, on the form of the action. It turns out that the only thing left to do then is, instead of computing complicated path integrals, to develop an intuition for working with the complex space describing physical vacua of the theory. This was done by Seiberg and Witten in [1,2].

This solution of the supersymmetric Yang-Mills (SYM) theories supplied a lot of material for future investigations. The main purpose of this work is to study the behavior of a certain class of supersymmetric states, namely, Bogomol'nyi-Prasad-Sommerfield (BPS) states, under the adiabatic variation of low energy moduli of the theory. BPS states are of crucial importance in SUSY theories, since their spectrum can be easily found. We shall focus on the $\mathcal{N}=2$ $SU(2)$ SYM theory with matter and mainly consider the BPS states in the sector with a nontrivial magnetic charge. As we are going to demonstrate, under a certain variation of the moduli, localized BPS states become delocalized. Predictions of such behavior can easily be obtained from the exact solution of the theory. They are actually presented in the original paper [2]. However, the explanation given there uses only general arguments (which we briefly review in the next section) and does not provide the explicit mechanism of such an unusual behavior. It was also noticed there that by treating the coupling constant as small one can examine this phenomenon in detail through considering classical solutions to the equations of motion. Our goal here is to satisfy this program and to demonstrate explicitly how this mechanism works.

According to Seiberg and Witten, to obtain the dual description of the theory, one should consider the variation of moduli, which corresponds to a closed loop on the moduli space around a singular point where the low energy description fails. If this loop is topologically trivial, the corresponding transformation is trivial and we obtain the same theory description. But if we encircle the singular point, we come to another description of the same theory. As we know from the famous example of electric-magnetic duality due to Montonen and Olive [3], a duality transformation may interchange the roles of electric and magnetic charges. Something similar happens in the above case. Seiberg and Witten proposed that the Montonen-Olive duality is only a special case of a more general group of duality transformations.

In the present work we shall consider the theory only in the vicinity of one type of singularity. These are the points where the quark becomes massless. Let us focus on the sector with unit magnetic charge. The topic of our main interest is the combined state consisting of a monopole and a fermion. Under certain conditions these two objects may coexist as a bound state, i.e., a configuration with localized energy. The set of points on the moduli space where delocalization necessarily occurs is called the curve of marginal stability (CMS). We shall refer to this delocalization as the "decay" of a bound state.[1]

It happens that the massless quark singularity belongs to a CMS. Thus, one will intersect the CMS twice in moving along the small closed loop, around it. At some part of the loop, one will have the monopole-fermion bound state; at another, this state will be unbound. On the other hand, since on closing the loop the theory description may switch to the dual one, one might expect that the charges will transform in a nontrivial way, such that the unbound state again becomes bound. This indeed takes place in the case we are going to consider.

As was mentioned above, we are going to investigate the process of bound state "decay" or delocalization by solving the classical equations of motion near the singular point, treating this region of the moduli space semiclassically. We analyze the solutions to the equations of motion and deter-

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[1] We use the terminology of [4], where similar physics was studied.

mine the energy spectrum of the fermion states in the field of the monopole. We also derive the restrictions on the existence of the bound state. Collecting all the pieces together, the shift of the charges due to a nontrivial duality transformation predicted by simple Seiberg-Witten simple arguments can be explicitly recovered.

The paper is organized as follows. We give more explicit grounds for the speculations above and present a brief introduction to the Seiberg-Witten method in Sec. II. The reader familiar with the topic may skip this introduction. In Sec. III we solve the classical equations for a fermion-monopole system and analyze the solution, thus describing the bound state decay semiclassically. Considering the transformation of the fermion spectrum during an encircling singularity, we reproduce the charge shift predicted by Seiberg and Witten. Section IV is the conclusion. We also give there several brief comments on the physics underlying the phenomenon.

This work has several common points with previous results and we refer the interested reader to the following papers. The conditions for existence of the bound state were first derived in [5]. The notion of BPS state decay was originally introduced in the work by Bilal and Ferrari [4], where this phenomenon was first derived. See [6] and [7], and references therein. Finally, the appropriate CMS was studied in [8].

II. PREHISTORIC

A. Discussion of Seiberg-Witten theory

In two papers in 1994, Seiberg and Witten managed to evaluate the exact low energy effective action for both pure $\mathcal{N}=2$ super Yang-Mills theory [1] and $\mathcal{N}=2$ super Yang-Mills theory with matter [2]. The method they applied was based on an investigation of the symmetries of the action rather than on evaluating functional integrals. It is well known that supersymmetry imposes severe restrictions on the form of the action. Thus, for $\mathcal{N}=2$ SYM theory the action is fixed up to some locally holomorphic function[2]

$$S_{N=2}=\frac{1}{16\pi}\mathrm{Im}\left[\mathrm{Tr}\int d^2\theta d^2\tilde{\theta}\mathcal{F}(\Psi)\right], \tag{1}$$

where $\mathcal{F}$ is the mentioned locally holomorphic function, called the prepotential, and Ψ denotes the $\mathcal{N}=2$ superfield. We stress that the action can have such a simple form only because of supersymmetry.

The classical form of $\mathcal{F}$ is fixed by renormalizability to be $\mathcal{F}=1/2\tau\Psi^2$, with τ the complex bare coupling constant[3]

$$\tau=\frac{\theta}{2\pi}+\frac{4\pi i}{g^2},$$

where θ is the QCD θ angle. The rather simple expression (1) actually encodes a long action containing the vector gauge field A_μ, two Weyl fermion fields ψ and λ, and one complex scalar field A in the adjoint representation of some gauge group G:

$$\begin{aligned}S_{YM}=\int d^4x\frac{1}{g^2}\bigg(&-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\frac{g^2\theta}{32\pi^2}F_{\mu\nu}\tilde{F}^{\mu\nu}-i\lambda\sigma^\mu D_\mu\bar{\lambda}\\&+\frac{1}{2}D^2+(D_\mu A)^\dagger D_\mu A-i\overline{\psi\sigma}^\mu D_\mu\psi-D[A^\dagger,A]\\&-i\sqrt{2}[\lambda,\psi]A^\dagger-i\sqrt{2}[\bar{\lambda},\bar{\psi}]A+F^\dagger F\bigg).\end{aligned} \tag{2}$$

As can be seen from Eq. (2), the action possesses a certain potential V arising after integrating out auxiliary fields. In supersymmetric theories the vacua of the theory are simply the zeros of the potential:

$$V=\frac{1}{2}\mathrm{tr}[A^\dagger,A]^2=0.$$

Since A belongs to the adjoint representation of a gauge group, the vacua of $\mathcal{N}=2$ theory are described by Cartan elements of Lie algebra of the group G. Further, we shall concentrate particularly on the case of the $SU(2)$ gauge group. For this case the vacua correspond, after appropriate gauge transformation, to values of A along the σ^3 direction:

$$\langle A\rangle=\frac{1}{2}a\sigma^3.$$

The complex quantity a parameterizes the space of classical vacua, called the moduli space of the theory.

Seiberg and Witten were searching for the low energy effective action. This action should contain only massless particles. Because of the Higgs effect most of the fields acquire masses. However, the components of non-Abelian fields along the σ^3 direction are still massless and therefore the effective description should be a functional of these fields. Evaluation of the effective action is generally a highly complicated task. However, in this case supersymmetry simplifies the problem essentially, imposing restrictions on the quantum action as well. The massless part of the action (1) fixed by supersymmetry in more familiar $\mathcal{N}=1$ notation has the form

$$S=\frac{1}{16\pi}\mathrm{Im}\left[\int d^2\theta\ \mathcal{F}''(\Phi)W^\alpha W_\alpha+\int d^4\theta\ \Phi^+\mathcal{F}'(\Phi)\right], \tag{3}$$

where W_α and Φ denote the $\mathcal{N}=1$ vector and chiral multiplets, respectively. Now to find the effective action we must determine the quantum form of the prepotential $\mathcal{F}$. This can be done by investigating the moduli space of the theory.

Classically, the moduli space is a complex plane with one fixed point $a=0$, where the broken symmetry is restored. We shall refer to this point as a singularity, since there the effective low energy description (3) breaks down. In the quantum

[2]For more detailed reviews of the $\mathcal{N}=2$ and $\mathcal{N}=1$ actions, see, for example, [9].

[3]Note that $\tau=\mathcal{F}''$. This relation holds in the quantum case as well.

case there are several additional singularities on the moduli space, where some massive modes become massless.

If we wrote the expression (3) in component fields, we should notice that the coefficient of the kinetic term for the scalar field A would be proportional to imaginary part of the second derivative of $\mathcal{F}$. In other words, the metric on the moduli space would be

$$ds^2 = \mathrm{Im}[\mathcal{F}''(a)]\, da\, d\bar{a}.$$

Being a harmonic function, this metric can be positive definite only in the case of a meromorphic function $\mathcal{F}$, and the singularities of $\mathcal{F}$ exactly coincide with the mentioned singularities of the moduli space.

Now let us introduce a new variable a_D which is the Legendre conjugate of a,

$$a_D = \frac{\partial \mathcal{F}(a)}{\partial a}. \tag{4}$$

This variable will naturally arise in the quantum generalization of the central charge formula. The central charge appears as the nontrivial commutator of SUSY generators, being a central extension of the algebra:

$$\{Q^I_\alpha, Q^J_\beta\} = \sqrt{2}\epsilon^{IJ}\epsilon_{\alpha\beta} Z.$$

For a given physical state, the central charge is connected with the mass of the state via the so-called Bogomol'nyi inequality [10]:

$$M^2 \geqslant 2|Z|^2.$$

The states that saturate this inequality are called BPS states [11]. The latter protect one-half of supersymmetry and their mass is equal to the central charge. The spectrum of these states can always be exactly determined since the supersymmetry preserves the Bogomol'nyi inequality in the quantum case as well. One needs only to determine the central charge for the corresponding state. The classical central charge for $\mathcal{N}=2$ SYM was calculated by Witten and Olive in [10]. It is generated by the vacuum expectation value of the scalar field and is naturally expressed in terms of the electric and magnetic charges of the physical state. The quantum formula for the central charge, introduced in [1], is supposed to be

$$Z = n_e a + n_m a_D. \tag{5}$$

This formula reveals the reciprocal character of a and a_D. In fact it can be analyzed from the action (3) that the Legendre transformation (4) generated by $\mathcal{F}$ leads to the same action in terms of the dual fields (Φ_D, W^α_D), with the new coupling $\tau_D = 1/\tau$. Obviously, the electric and magnetic charges interchange under this transformation, but the central charge (5) stays invariant, since it is connected to the observable quantity, the mass of the state. This is the famous Montonen-Olive electric-magnetic duality. Seiberg and Witten found that the full duality group is larger, namely, all $SL(2,\mathbb{Z})$ transformations leave the theory invariant.

As soon as every point of the moduli space is characterized by a pair (a, a_D), there is a spectrum of BPS states attached to each point. These states are characterized by definite electric and magnetic charges. Seiberg and Witten proposed considering the central charge formula (5) as the product of a vector consisting of a_D and a and a row of charges (n_m, n_e). This form reflects the idea that a_D and a belong to a representation of the duality group $SL(2,\mathbb{Z})$ and helps to understand how the dualities or monodromies rotate the charges of the physical state.

To determine the monodromies one could consider the low energy theory in the vicinity of singularities and calculate the perturbative one-loop β function, with a playing the role of scale:

$$\beta(g) \equiv a\frac{dg^2}{da}. \tag{6}$$

The latter can be established from consideration of the spectrum of the theory at the singular point. Indeed, the prepotential defines the coupling constant $\tau = \mathcal{F}''$, that is, $\tau = \partial a_D/\partial a$. Derived from Eq. (6), the behavior of a_D as a function of a, while encircling the singularity, fixes the monodromy. This will be done in detail in the next subsection. Possessing the information about the number of singularities and their monodromies, it is possible to completely restore the $SL(2,\mathbb{Z})$ fiber bundle (a_D, a) over the moduli space and thus the functions $\mathcal{F}$ and τ. This is enough for recovering the explicit form of the effective action; however, we do not go into this any further here.

B. Theory with matter

Consideration of the theory with matter implies several slight modifications. On the level of the Lagrangian one adds a term which corresponds to N_f $\mathcal{N}=2$ hypermultiplets or N_f pairs of $\mathcal{N}=1$ chiral and antichiral multiplets, in either adjoint or fundamental representation. We shall be interested in the choice of fundamental matter and the number of flavors being 1. From the $\mathcal{N}=1$ point of view we add one chiral and one antichiral multiplet, each consisting of a complex scalar field and a Weyl fermion. The full Lagrangian in $\mathcal{N}=1$ notation has the form

$$\begin{aligned} S = \frac{1}{8\pi}\mathrm{Im}\,\mathrm{Tr}\Bigg[\tau\Bigg(\int d^4 x d^2\theta W^\alpha W_\alpha \\ + 2\int d^4x d^2\theta d^2\bar{\theta}\Phi^\dagger e^{-2V}\Phi\Bigg)\Bigg] \\ + \int d^4x d^2\theta d^2\bar{\theta}(Q_i^\dagger e^{-2V}Q_i + \tilde{Q}_i e^{2V}\tilde{Q}_i^\dagger) \\ + \Bigg(\int d^4x d^2\theta\sqrt{2}\tilde{Q}_i\Phi Q_i + m_i\tilde{Q}_i Q_i + \mathrm{H.c.}\Bigg), \end{aligned} \tag{7}$$

and its matter part in components looks like

$$S_{matter}=\int d^4x(D_\mu q)^\dagger D_\mu q+D_\mu\tilde{q}(D_\mu\tilde{q})^\dagger-i\overline{\psi_q\sigma^\mu}D_\mu\psi_q$$
$$+i\psi_{\tilde{q}}\sigma^\mu D_\mu\overline{\psi_{\tilde{q}}}-q^\dagger Dq+\tilde{q}D\tilde{q}^\dagger-i\sqrt{2}q^\dagger\lambda\psi_q$$
$$+i\sqrt{2}\tilde{q}\overline{\lambda\psi_{\tilde{q}}}+i\sqrt{2}\overline{\psi_q\lambda}q-i\sqrt{2}\psi_{\tilde{q}}\lambda\tilde{q}^\dagger+F_q^\dagger F_q$$
$$+F_{\tilde{q}}F_{\tilde{q}}^\dagger+[\sqrt{2}\tilde{q}AF_q+\sqrt{2}F_{\tilde{q}}Aq-\sqrt{2}\psi_{\tilde{q}}A\psi_q$$
$$-\sqrt{2}\tilde{q}\psi\psi_q-\sqrt{2}\psi_{\tilde{q}}\psi q+\tilde{q}Fq+m\tilde{q}F_q+mF_{\tilde{q}}q$$
$$-m\psi_{\tilde{q}}\psi_q+\text{H.c.}].$$

The matter part of the action possesses $U(N_f)$ global symmetry. The $\mathcal{N}=1$ mass term generally breaks this symmetry down to a product of N_f copies of $U(1)$. Therefore the matter fields carry additional global charges with respect to this $U(1)$'s. We might assume that these charges also contribute to the central extension of the SUSY algebra, and it is indeed the case: the central charge formula (5) is modified in the following way:

$$Z=n_e a+n_m a_D+\sum_{N_f}\frac{m_i}{\sqrt{2}}S_i, \tag{8}$$

where the S_i's are the corresponding $U(1)$ global charges.

Below we are going to concentrate on a particular type of singularity of the moduli space, namely, the ones corresponding to the massless quarks. If we look at the central charge formula, we shall see that the BPS states with charges $(0,-S,S)$ become massless at the point of the moduli space $a_0=m/\sqrt{2}$. At this point the matter fields (quarks and scalars) become massless. This fact could also be easily derived from the classical Lagrangian (2). Since the spectrum of the low energy effective theory is known, the one-loop β function can be calculated in the vicinity of this point:

$$\beta(g)=\frac{g^3}{8\pi^2}.$$

This β function is exact perturbatively since the higher loop corrections are forbidden by supersymmetry. Furthermore, since the nonperturbative part is regular in the vicinity of the singularity we can simply drop it. From the β function we restore the leading logarithmic part of the coupling τ as a function of the scale a and next the leading order of the function a_D, which has the following form near the point a_0:

$$a_D\simeq c-\frac{i}{2\pi}(a-a_0)\ln(a-a_0). \tag{9}$$

For the theory with matter, the central charge (8) is represented as a product of the row of charges (n_m,n_e,S_f) and a column consisting of a_D, a, and m. The expansion (9) allows one to investigate the monodromies of a_D around the singularity. Performing a closed contour around a_0 in the a plane gives the following transformation law for a_D:

$$a_D\to a_D+a-\frac{m}{\sqrt{2}}.$$

This transformation law can be written as the matrix acting on the vector (a_D,a,m):

$$M=\begin{pmatrix}1&1&-1\\0&1&0\\0&0&1\end{pmatrix}.$$

What happens to the central charge after the encircling singularity? Seiberg and Witten claimed that the latter should not change under the monodromy that arises, since it is connected to the observable mass of the BPS state. The central charge invariance implies that the row of charges must also be transformed. Apparently, it should be rotated by the inverse monodromy matrix M^{-1}. Thus, we find the charge transformation law:

$$(n_m,n_e,S)\to(n_m,n_e-n_m,S+n_m). \tag{10}$$

Looking at this transformation law, we notice that by encircling the singularity multiple times, in the presence of a monopole, we can make the S charge arbitrarily large. Let us concentrate on studying the BPS states consisting of a monopole and quarks. Quarks, being the matter fields, possess the S charge. This charge simply counts the fermion number of the physical state. However, since the central charge is invariant and so is the mass or the energy of the corresponding BPS state, the number of quarks that are sitting on the same energy level, if the latter is quantized, is fixed according to the Pauli exclusion principle.[4] Thus, on the one hand the fermion number is fixed, but on the other hand it can be made arbitrarily large. This paradox may be resolved by assuming that the additional fermion arising after encircling the singularity leaves the quantized energy level and falls away from the monopole.

In the limit $m\gg\Lambda$ $(g^2\to 0)$, where Λ is the dynamically generated scale of the theory, the singularity corresponding to massless quark becomes semiclassical. This means that the semiclassical approach is reliable for an investigation of the behavior of a quark-monopole bound state. Below we solve the classical equations for the fermion mode in the monopole background configuration. Studying the properties of the solution, we observe the variation of the BPS spectrum when encircling the singularity, which is consistent with the observations of the charge shift made above.

III. DERIVATION OF THE FERMION MODE

A. Solution to Dirac equation

Consider the $SU(2)$ theory with the action (7) in the semiclassical part of the moduli space $a\gg\Lambda$. Assume that the point $a=m/\sqrt{2}$ also belongs to this region. The theory

[4]As we shall see later, there is one separate fermionic energy level in the presence of a monopole.

contains monopoles in its spectrum. There are also solutions that correspond to matter fermions in the external monopole field. The bound state corresponds to the configuration with localized energy. Therefore the corresponding fermion mode should be normalizable. As we shall see, such normalizable solutions to the classical equations exist only in certain regions of the moduli space.

The fermionic components of the hypermultiplet satisfy the Dirac equation

$$i\gamma^\mu D_\mu \Psi - (\hat{m} + \sqrt{2}\hat{A})\Psi = 0,$$

$$\hat{m} + \sqrt{2}\hat{A} = \mathrm{Re}(m + \sqrt{2}A) + i\gamma^5 \mathrm{Im}(m + \sqrt{2}A), \tag{11}$$

where the Dirac spinor Ψ is composed of two Weyl spinors as follows:

$$\Psi = \begin{pmatrix} \psi_q \\ \overline{\psi_{\tilde{q}}} \end{pmatrix}.$$

These two Weyl spinors belong to fundamental and antifundamental $\mathcal{N}=1$ chiral multiplets.

Working in the Gauss gauge $A_0 = 0$, substitute D_0 by the energy eigenvalue iE. Now notice that the Hamiltonian commutes with the operator $\Gamma = \gamma^0\gamma^5$ in the case $\mathrm{Im}(m) + \sqrt{2}\,\mathrm{Im}(A) = 0$. Thus a general solution to Eq. (11) can be found as the sum of Γ eigenfunctions $\Psi = \Psi^+ + \Psi^-$.

Substituting Ψ^+ instead of Ψ we split Eq. (11):

$$[\sigma^k D_k - \mathrm{Re}(m + \sqrt{2}A)]\psi^+ = 0,$$

$$[E - \mathrm{Im}(m + \sqrt{2}A)]\psi^+ = 0. \tag{12}$$

Here ψ^+ is the chiral Weyl component of Ψ.

In the consideration above we treat m as a real constant in contrast with the scalar field A which is generally complex. However, it would be more useful for future computations to keep A real. Since there is a $U(1)_{\mathcal{R}}$ symmetry acting on the fields, we could choose a $U(1)$ rotation

$$A \to e^{2i\alpha}A,$$

$$m \to e^{2i\alpha}m,$$

$$\Psi \to e^{-i\alpha\gamma^5}\Psi,$$

under which the complexity of A flows into the complexity of m. Although this symmetry is in general anomalous, this is not the case in the semiclassical region. Then the solution to the second equation of (12) implies that the bound state energy spectrum satisfies

$$E = \mathrm{Im}(\mathcal{M}),$$

where $\mathcal{M}$ now defines a complex mass parameter.

In the case of a real scalar field A we can use the standard radially symmetric solution for the 't Hooft–Polyakov monopole [12] in the BPS limit [11]:[5]

$$\sqrt{2}A^i = a n^i [1 - F(r)],$$

$$A_i^a = \epsilon^{aij}\frac{n^j}{r}[1 - H(r)], \tag{13}$$

where n^i is a unit vector, and F and H are known functions of radius r. For future convenience we chose the normalization of the scalar field A in Eq. (13) slightly different from that in [2]. In our convention asymptotically $A \to a/\sqrt{2}$. Furthermore, in [2] the charges of the BPS states were renormalized in such a way that $a \to a/2$. So our a will be different from that of [2] by the factor of $(2\sqrt{2})^{-1}$.

Substituting Eq. (13) into Eq. (12) we find a solution in the form

$$(\psi^+)^\alpha_{\ a} = \delta^\alpha_a \chi_0 \xi + n^i(\sigma^i)^\alpha_{\ \beta}\delta^\beta_{\ a}\eta_0\zeta, \tag{14}$$

where α and a are spinor and color indices, respectively. The functions χ_0 and η_0 solving the equations at zero m are defined as follows:[6]

$$\chi_0 = \frac{1}{\sqrt{\rho\sinh\rho}}\tanh\frac{\rho}{2},$$

$$\eta_0 = \frac{1}{\sqrt{\rho\sinh\rho}}\coth\frac{\rho}{2}. \tag{15}$$

The functions ξ and ζ satisfy the system of first order differential equations

$$\xi' = r_0 m \zeta \coth^2\left(\frac{\rho}{2}\right),$$

$$\zeta' = r_0 m \xi \tanh^2\left(\frac{\rho}{2}\right), \tag{16}$$

which implies the pair of second order equations:

$$\xi'' + \frac{2\xi'}{\sinh\rho} - r_0^2 m^2 \xi = 0,$$

$$\zeta'' - \frac{2\zeta'}{\sinh\rho} - r_0^2 m^2 \zeta = 0. \tag{17}$$

Explicit solutions to these equations can be found in terms of hypergeometric functions. We shall find normalizable solutions after an analysis of the asymptotic behavior of the general solutions to Eq. (12).

[5]We neglect the back reaction of matter fields on the monopole fields, assuming that in the semiclassical limit the ratio of the corresponding masses is small.

[6]Here we define $\rho = r/r_0$, $r_0^{-1} = a$.

The solution presented above is one of positive chirality. It can be shown as well that for negative chirality a similar solution cannot be made normalizable. It is also natural to assume that there are no other discrete spectrum solutions to Eq. (11). See the discussion in [5].

B. Analysis of the solution

In this subsection we investigate the asymptotic of the solution to Eq. (11) and rederive the conditions under which this solution exists [5]. Using the asymptotic we also derive the explicit solution in terms of hypergeometric functions.

We introduce a pair of functions χ,η that will characterize the asymptotic behavior of the solution (14):

$$\chi=\chi_0\xi, \qquad \eta=\eta_0\zeta.$$

Since we are interested in soliton-fermion bound states, the fermion mode we are looking for should be a normalizable one, i.e., it should decrease fast enough at infinity and be regular at the origin, namely,

$$\chi(0)=\text{const}, \qquad \eta(0)=0.$$

In the $\rho\to 0$ limit the fermion mode behaves as the solution to the equations

$$\xi''+\frac{2\xi'}{\rho}-r_0^2m^2\xi=0,$$

$$\zeta''-\frac{2\zeta'}{\rho}-r_0^2m^2\zeta=0. \tag{18}$$

The regular solutions to Eq. (18) are as follows:

$$\xi=C\frac{\sinh(r_0m\rho)}{\rho},$$

$$\zeta=C(r_0m\rho-1)e^{r_0m\rho}+C(r_0m\rho+1)e^{-r_0m\rho}, \tag{19}$$

with the coefficients fixed by the asymptotic behavior of Eq. (16).

Assume that we have a solution that is regular at the origin. From the asymptotic (19) it follows that the first derivative of this solution is zero and the second derivative is positive at this point. Furthermore, the second derivative in the exact equations (17) is positive wherever the first derivative vanishes. Therefore such solutions can have only minima and cannot be normalizable. This means that the solution to Eq. (17) is either regular at the origin and divergent at infinity or regular at infinity and divergent at the origin. However, if we look at Eq. (15), we shall see that under certain conditions we can construct a normalizable zero mode. The solution to (17) regular at the origin and increasing at infinity has the asymptotic

$$\xi,\zeta\sim e^{r_0m\rho}.$$

It follows from Eq. (15) that the functions χ_0,η_0 have the asymptotic

$$\chi_0,\eta_0\sim\frac{e^{-\rho}}{\sqrt{\rho}},$$

and that for the functions χ,η:

$$\chi,\eta\sim\sqrt{\frac{1}{\rho}}\exp\left[\left(r_0m-\frac{1}{2}\right)\rho\right].$$

We see that for the existence of the normalizable zero mode the following conditions should be satisfied:

$$2r_0m<1 \quad \text{or} \quad \frac{2m}{a}<1,$$

or $a>m/\sqrt{2}$ in the standard normalization of [2]. This result was originally derived in [5], and then was confirmed by CMS considerations in [8] in the weak coupling limit.

Now turn to Eqs. (17). Changing variables from ρ to $x=(\cosh\rho+1)/2$ we obtain a pair of hypergeometric equations:

$$x(x-1)y_1''+\left(x+\frac{1}{2}\right)y_1'-r_0^2m^2y_1=0,$$

$$x(x-1)y_2''+\left(x-\frac{3}{2}\right)y_2'-r_0^2m^2y_2=0. \tag{20}$$

Solving Eq. (20) and expressing the solution in terms of the original variable ρ we find the unknown functions ξ and ζ:

$$\xi(\rho)=CF\left(\frac{m}{a},-\frac{m}{a},-\frac{1}{2},\cosh^2\frac{\rho}{2}\right),$$

$$\zeta(\rho)=\tilde{C}F\left(\frac{m}{a},-\frac{m}{a},\frac{3}{2},\cosh^2\frac{\rho}{2}\right).$$

Here we took into account the asymptotic of the solution determined above. The asymptotic of Eq. (16) also fixes the relation $C=\tilde{C}$.

C. BPS state decay

From the considerations above we found that the normalizable fermion mode with the energy defined by the complex mass parameter $E=\operatorname{Im}\mathcal{M}$ exists only in the region of moduli space defined by[7]

$$\operatorname{Re}\mathcal{M}=m<\frac{|a|}{2}. \tag{21}$$

Now for a complete understanding of the process we need one more thing, namely, to find what region of the moduli space corresponds to the solutions to the Dirac equation (11)

[7] In the fermion equations a was treated as the vacuum expectation value of the real field A. To define this condition for all complex moduli plane we substitute the latter by the absolute value of the complex coordinate $|a|$.

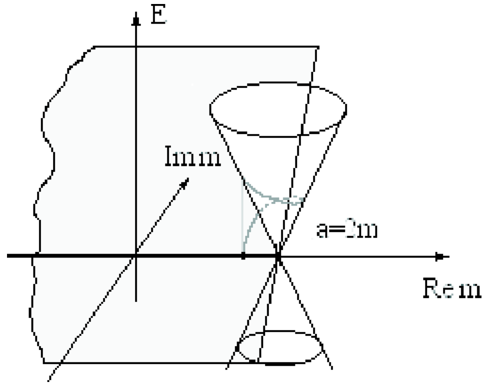

FIG. 1. The region of discrete spectrum (inclined semiplane) and the region of continuous spectrum (interior of the cone) intersect via the line $E = \mathrm{Im}\,\mathcal{M}$ which goes through the point $a = 2m$.

from the continuous spectrum, i.e., nonbound BPS states. Considering the asymptotic of Eq. (11) at infinity, we obtain a system of linear differential equations, neglecting exponentially decaying nonconstant coefficients. The corresponding characteristic equation gives for fixed energy E

$$E = \pm \left| m - \frac{a}{2} \right|.$$

This condition defines a cone in the E-a or equivalently in the E-$\mathcal{M}$ parameter space, with the singularity at $a = 2m$ (Fig. 1). Further, we substitute the investigation on the complex a plane by that on the $\mathcal{M}$ plane, which is equivalent because of the $U(1)_{\mathcal{R}}$ symmetry. The continuous spectrum belongs to the interior of this cone. The region of the discrete spectrum is an inclined plane $E = \mathrm{Im}\,\mathcal{M}$, which is bounded due to the condition (21).

Now a closed contour on the complex $\mathcal{M}$ plane corresponds to some three-dimensional curve, which is restricted to belong either to an inclined plane of the discrete spectrum or to the interior of the continuous spectrum cone, where, we suppose, are the only possible loci for a consistent physical state. Since the contour must be closed and smooth, this curve must have a spiral form. This means that on encircling the singular point we cannot return to the initial value of the energy. Furthermore, the final point of the spiral curve belongs to a state from the continuous spectrum, which means a decay of the initial bound state.

To clarify what happens when we encircle the singularity, consider the dependence of the energy E on the angle of rotation φ in the $\mathcal{M}$ plane (Fig. 2). We start at some real mass value m_0 and hence at zero energy. At $\varphi = \pi/2$ we reach the continuous spectrum. For φ from $\pi/2$ to $3\pi/2$ there is no bound state, i.e., the initial bound state has decayed. After a full rotation $\varphi = 2\pi$ the mode from the continuous spectrum cannot descend to a discrete spectrum and the only allowed values of energy will be $E \geqslant |m_0 - a/2|$. However, if we move in the opposite direction, starting from $\varphi = 2\pi$, we shall also come to the continuous spectrum at $\varphi = 3\pi/2$, but in the part of it which is below the zero level. Since we move adiabatically, and all the states for fermions below zero are occupied, we conclude that after encircling the singularity one fermion mode runs to the continuous spectrum, and another comes from the Dirac sea below.

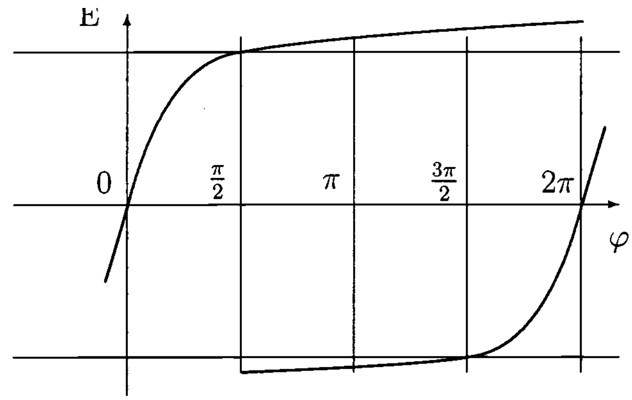

FIG. 2. The initial bound state decays after crossing the point of marginal stability at $\varphi = \pi/2$. Another fermion mode replaces it, rising from the continuous spectrum below.

IV. CONCLUSIONS

We found and investigated solutions to the classical equations of motion (11). We were searching for solutions with localized energy or, in other words, normalizable solutions. We have established that this kind of field configuration exists only in a definite region of the moduli space. Our interest was in the process of encircling a singular point. We found that, starting from the bound state and closing a loop around the singularity, one comes to a multiparticle state consisting of a bound state of a fermion and a monopole, and a distant delocalized fermion mode in the continuous part of the energy spectrum.

This result was expected from Seiberg-Witten considerations of the moduli space of the $\mathcal{N} = 2$ theory. Considering just the BPS mass formula (8) and claiming the invariance of the central charge under monodromy, it was stated that the charges should transform as follows:

$$(n_m, n_e, S) \rightarrow (n_m, n_e - n_m, S + n_m).$$

This is indeed the case in a semiclassical analysis with charges $(1, -1, 1)$, namely, the magnetic charge is conserved, and the electric charge is shifted by 1, due to the fermion mode in the continuous spectrum. The S charge also accounts for this additional fermion mode.

What is the physical meaning of this result? Strictly speaking we do not discuss the process of some observable decay of a physical state. The original word *decay* may sound slightly misleading, because we are concerned with *delocalization* rather than decay. The process of encircling a singularity does not seem to be very physical, since the motion around the singular point takes place in the abstract moduli space. However, there is strong evidence that the moduli spaces of various supersymmetric theories are related to the phase spaces of certain dynamical systems. (For a discussion and references, see [13].) Not taking into account this relation, the motion around the singularity actually corresponds to the duality transformation, or transition from one theory description to another. We are familiar with special cases of this transition.

Consider the electric-magnetic duality. According to the Dirac quantization rule, a large value of electric charge e corresponds to a small value of magnetic charge $1/e$. When the electric charge becomes large, it is nonsensical to use the perturbative expansion with the electric charge as the parameter. The dual theory with the small magnetic charge may be used instead. In fact we are still working with the same theory, but use another description for it. According to this description, the fields that were electrically charged look magnetically charged after the transition.

In the above example we consider a simple duality transformation. From the point of view of Seiberg-Witten theory, the latter case corresponds to the real scale (one-dimensional moduli space) and is only a subgroup of the duality group. In this paper we worked with the complex moduli space, and therefore a complex scale. Due to the complexity, multivaluedness appeared. This multivaluedness naturally described the existing variety of descriptions of the theory on a complex moduli space. We have been convinced that for the case of $\mathcal{N}=2$ SYM theory with matter, it is necessary to take into account the whole set of charges (n_m, n_e, S) for correct application of duality transformations.

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